

Equality of Proofs for Linear Equality

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Abstract

This paper is about equality of proofs in which a binary predicate formalizing properties of equality occurs, besides conjunction and the constant true proposition. The properties of equality in question are those of a preordering relation, those of an equivalence relation, and other properties appropriate for an equality relation in linear logic. The guiding idea is that equality of proofs is induced by coherence, understood as the existence of a faithful functor from a syntactical category into a category whose arrows correspond to diagrams. Edges in these diagrams join occurrences of variables that must remain the same in every generalization of the proof. It is found that assumptions about equality of proofs for equality are parallel to standard assumptions about equality of arrows in categories. They reproduce standard categorial assumptions on a different level. It is also found that assumptions for a preordering relation involve an adjoint situation.

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1 Introduction

The purpose of this paper is to investigate in equational logic the hypothesis that two proofs are equal if and only if they have the same generality. Two proofs,

with the same premises and conclusions, have the same generality when after diversifying variables as much as possible without changing the rules of inference one ends up again with the same premises and conclusions, up to renaming of variables. This notion of generality of proof was investigated in [6] (see also [7]), [8] and [9]. However, individual variables (as opposed to propositional variables) and equality between them are mentioned only briefly, as an example, in [6] (end of Section 3). The present paper develops more fully these matters announced in that previous paper.

The results of this paper take the form of coherence theorems, understood as faithfulness results for functors from syntactically constructed categories to a category whose arrows correspond to diagrams. Edges in these diagrams join occurrences of variables that must remain the same in every generalization of the proof.

The central result of the paper is that assumptions about equality of proofs for equality induced by generality are parallel to standard assumptions about equality of arrows in categories. Usual categorial assumptions are reproduced on a different level. Reflexivity of equality corresponds to identity arrows, and transitivity corresponds to categorial composition of arrows. Equations involving reflexivity and transitivity are parallel to the categorial assumptions of omission of identity arrows in composition and associativity of composition. Similar correspondences hold for other postulates concerning equality.

Another result of the paper is that assumptions for a preordering relation involve an adjoint situation. This was foreshadowed in [14].

Besides [14], another paper about equality of proofs for equality is [12], which investigates the equivalence of various syntactical formulations in classical predicate logic with equality. There is a chapter on equational logic based on fibrations in [11] (Chapter 3), but, contrary to what we have in this paper, it is asserted there (p. 174) that there are no different proofs of the same proposition.

We restrict ourselves to equality added to multiplicative conjunctive propositional linear logic, with the multiplicative constant true proposition. We eschew going beyond this limited fragment of equational logic because generality of proofs involving equality, like that of proofs involving implication, prevents the arrows corresponding to the structural rules of contraction and thinning from making natural transformations (cf. [5], Section 1, [8], Section 14.3, and [10]). This requirement of naturality is otherwise quite natural, and proof-theoretically well motivated (see [8], Chapters 9-11). Contraction and thinning are required if we want to say that we deal with full equational logic. In their absence, we cannot pretend to cover more than a relation of equality appropriate for linear logic (such as the equality relations investigated in [1]). This explains the expression *linear equality* in the title of the paper.

We restrict ourselves in general to a context with minimal assumptions where our results can be obtained. So we do not assume the commutativity of multiplicative conjunction if this is not essential; i.e., we work also in noncommutative linear logic (which is related to the Lambek calculus; we stay however within

the multiplicative conjunctive fragment of this calculus).

As we indicated above concerning adjunction, our results are about relations more general than equality relations, such as preordering and equivalence relations. Most of the paper (Sections 2-7) is about such relations. Only in the last section we indicate how to deal with further assumptions, such as congruence. Within our syntactical systems, where the means of expression are limited, the equivalence relations involved amount however to equality relations. In the last section our proofs will be less formal.

We consider binary operational expressions in the last section, but we do not have anywhere predicate variables. So we cannot say that we deal yet with full linear equational logic, but only with fragments of it. In the context where we investigate equality of proofs involving equality, motivated by generality, we would have to enter into the question of what is a linear predicate (see [2]), and moreover we would have to restrict ourselves to the multiplicative fragment of linear logic without propositional constants (see [9]). Equations that can be expected in that context would be on the lines of [12] (Sections 2.3-4). We leave however these extensions of our approach for another occasion.

2 The category M_{\leq}

Let \mathcal{V} be a set whose elements, for which we use the letters x, y, z, \dots , perhaps with indices, are called *variables*. The cardinality of \mathcal{V} is not restricted: \mathcal{V} can be infinite or finite, and even empty. Let words of the form $x \leq y$ or \top be called *atomic formulae*. The set of *formulae* is defined inductively as follows. Atomic formulae are formulae, and if A and B are formulae, then $(A \wedge B)$ is a formula. We use A, B, C, \dots for formulae, and we omit, as usual, the outermost parentheses of $(A \wedge B)$. (We proceed analogously for other similar expressions later on.)

The objects of the category M_{\leq} are formulae. To define the arrows of M_{\leq} , we define first inductively the *arrow terms* of M_{\leq} in the following way. We use f, g, h, \dots , perhaps with indices, for arrow terms. Every arrow term f has a *type*, which is an ordered pair (A, B) of objects of M_{\leq} ; that f is of type (A, B) is written $f: A \vdash B$.

For all formulae A, B and C , and for all variables x, y and z , the following are *primitive arrow terms* of M_{\leq} :

$$\begin{aligned} \mathbf{1}_A: A &\vdash A, \\ b_{A,B,C}^{\rightarrow}: A \wedge (B \wedge C) &\vdash (A \wedge B) \wedge C, \quad b_{A,B,C}^{\leftarrow}: (A \wedge B) \wedge C &\vdash A \wedge (B \wedge C), \\ \delta_A^{\rightarrow}: A \wedge \top &\vdash A, & \delta_A^{\leftarrow}: A \vdash A \wedge \top, \\ \sigma_A^{\rightarrow}: \top \wedge A &\vdash A, & \sigma_A^{\leftarrow}: A \vdash \top \wedge A, \\ r_x: \top &\vdash x \leq x, \\ t_{x,y,z}: x \leq y \wedge y \leq z &\vdash x \leq z. \end{aligned}$$

If $f: A \vdash B$ and $g: C \vdash D$ are arrow terms of M_{\leq} , then $f \wedge g: A \wedge C \vdash B \wedge D$ and $g \circ f: A \vdash D$ are arrow terms of M_{\leq} , provided that for $g \circ f$ we have that B and C are the same formula. This defines the arrow terms of M_{\leq} .

The arrows of M_{\leq} are equivalence classes of arrow terms with respect to the smallest equivalence relation which guarantees that the following equations are satisfied:

categorial equations:

$$\begin{aligned} (\text{cat 1}) \quad & \mathbf{1}_B \circ f = f, \quad f \circ \mathbf{1}_A = f, \quad \text{for } f: A \vdash B, \\ (\text{cat 2}) \quad & (h \circ g) \circ f = h \circ (g \circ f), \end{aligned}$$

bifunctioniality equations:

$$\begin{aligned} (\wedge 1) \quad & \mathbf{1}_A \wedge \mathbf{1}_B = \mathbf{1}_{A \wedge B}, \\ (\wedge 2) \quad & (g_1 \circ f_1) \wedge (g_2 \circ f_2) = (g_1 \wedge g_2) \circ (f_1 \wedge f_2), \end{aligned}$$

naturality equations:

for $f: A \vdash D$, $g: B \vdash E$ and $h: C \vdash F$,

$$\begin{aligned} (b \text{ nat}) \quad & ((f \wedge g) \wedge h) \circ b_{A,B,C}^{\rightarrow} = b_{D,E,F}^{\rightarrow} \circ (f \wedge (g \wedge h)), \\ (\delta \text{ nat}) \quad & f \circ \delta_A^{\rightarrow} = \delta_D^{\rightarrow} \circ (f \wedge \mathbf{1}_{\top}), \\ (\sigma \text{ nat}) \quad & f \circ \sigma_A^{\rightarrow} = \sigma_D^{\rightarrow} \circ (\mathbf{1}_{\top} \wedge f), \end{aligned}$$

specific equations of monoidal categories:

$$\begin{aligned} (bb) \quad & b_{A,B,C}^{\leftarrow} \circ b_{A,B,C}^{\rightarrow} = \mathbf{1}_{A \wedge (B \wedge C)}, \quad b_{A,B,C}^{\rightarrow} \circ b_{A,B,C}^{\leftarrow} = \mathbf{1}_{(A \wedge B) \wedge C}, \\ (b 5) \quad & b_{A \wedge B, C, D}^{\rightarrow} \circ b_{A, B, C \wedge D}^{\rightarrow} = (b_{A, B, C}^{\rightarrow} \wedge \mathbf{1}_D) \circ b_{A, B \wedge C, D}^{\rightarrow} \circ (\mathbf{1}_A \wedge b_{B, C, D}^{\rightarrow}), \\ (\delta \delta) \quad & \delta_A^{\leftarrow} \circ \delta_A^{\rightarrow} = \mathbf{1}_{A \wedge \top}, \quad \delta_A^{\rightarrow} \circ \delta_A^{\leftarrow} = \mathbf{1}_A, \\ (\sigma \sigma) \quad & \sigma_A^{\leftarrow} \circ \sigma_A^{\rightarrow} = \mathbf{1}_{\top \wedge A}, \quad \sigma_A^{\rightarrow} \circ \sigma_A^{\leftarrow} = \mathbf{1}_A, \\ (b \delta \sigma) \quad & b_{A, \top, C}^{\rightarrow} = (\delta_A^{\leftarrow} \wedge \mathbf{1}_C) \circ (\mathbf{1}_A \wedge \sigma_C^{\rightarrow}), \end{aligned}$$

specific equations of M_{\leq} :

$$\begin{aligned} (rt\delta) \quad & t_{x,y,y} \circ (\mathbf{1}_{x \leq y} \wedge r_y) = \delta_{x \leq y}^{\rightarrow}, \\ (rt\sigma) \quad & t_{y,y,x} \circ (r_y \wedge \mathbf{1}_{y \leq x}) = \sigma_{y \leq x}^{\rightarrow}, \\ (tb) \quad & t_{x,y,u} \circ (\mathbf{1}_{x \leq y} \wedge t_{y,z,u}) = t_{x,z,u} \circ (t_{x,y,z} \wedge \mathbf{1}_{z \leq u}) \circ b_{x \leq y, y \leq z, z \leq u}^{\rightarrow}; \end{aligned}$$

if $f_1 = g_1$ and $f_2 = g_2$, then $f_1 \wedge f_2 = g_1 \wedge g_2$ and $f_2 \circ f_1 = g_2 \circ g_1$, provided $f_2 \circ f_1$ and $g_2 \circ g_1$ are defined.

The category M_{\leq} is a monoidal category in the sense of [16] (Section VII.1, see also [8], Section 4.6). The equations $(rt\delta)$ and $(rt\sigma)$ are parallel to the equations (cat 1) , and the equation (tb) is parallel in the same manner to the equation (cat 2) . The equation (tb) , for instance, says that a composition tied to the arrows t is associative.

The following instance of (tb) :

$$t_{x,x,x} \circ (\mathbf{1}_{x \leq x} \wedge t_{x,x,x}) = t_{x,x,x} \circ (t_{x,x,x} \wedge \mathbf{1}_{x \leq x}) \circ b_{x \leq x, x \leq x}^{\rightarrow}$$

is analogous to the equation $(\check{b}\check{w})$ of categories with coproducts (see [8], List of Equations), where $t_{x,x,x}: x \leq x \wedge x \leq x \vdash x \leq x$ corresponds to the codiagonal arrow $\check{w}_p: p \vee p \vdash p$ of these categories. The instances of $(rt\delta)$ and $(rt\sigma)$ with x and y the same variable correspond in an analogous manner to the equations $(\check{w}\check{k})$ of categories with coproducts (see [8], ibid.). The arrow $r: \top \vdash x \leq x$ corresponds here to the arrow $\check{\kappa}_p: \perp \vdash p$ of categories with coproducts, \perp being the initial object.

The arrows r_x and $t_{x,y,z}$ codify of course the reflexivity and transitivity of a relation corresponding to \leq . So we deal here with a *preordering* relation.

3 The coherence of M_{\leq}

For every object A of M_{\leq} let GA be the number of occurrences of variables in A . (One could modify the category Br , mentioned below, so that its objects are formulae, instead of finite ordinals. In that case the object GA would be the formula A . We draw diagrams in this paper in that spirit. The category Br abstracts from a formula just the position of occurrences of variables in a formula, which is the only thing relevant for drawing diagrams. We do not expect the functor G below to be one-one on objects. It suffices for our purposes that it be faithful.)

Let us assign the following diagrams to the primitive arrow terms of M_{\leq}

$$\begin{array}{c} A \\ \mathbf{1}_A \\ A \end{array}$$

(where the line joining the two A 's stands for a family of parallel lines—one line for each occurrence of a variable in A ; for example, for A being $x \leq y \wedge z \leq x$ we have

$$\begin{array}{c} x \leq y \wedge z \leq x \\ | \quad | \quad | \\ x \leq y \wedge z \leq x \end{array}$$

and analogously in other cases below)

$$\begin{array}{ccc} b_{A,B,C}^{\rightarrow} & \begin{array}{c} A \wedge (B \wedge C) \\ | \quad | \quad | \\ (A \wedge B) \wedge C \end{array} & b_{A,B,C}^{\leftarrow} & \begin{array}{c} (A \wedge B) \wedge C \\ | \quad | \quad | \\ A \wedge (B \wedge C) \end{array} \end{array}$$

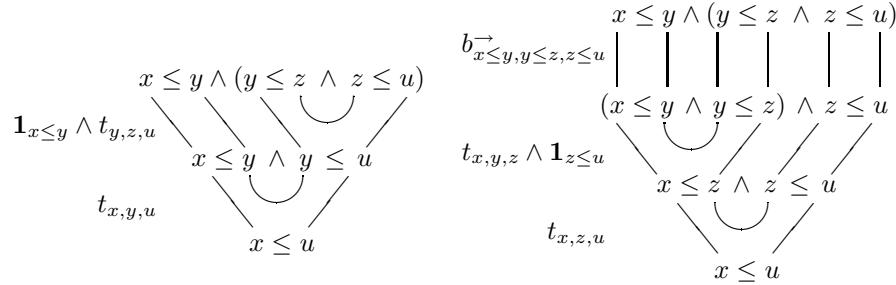
$$\begin{array}{ccc}
& A \wedge \top & \\
\delta_A^{\rightarrow} & \downarrow & \delta_A^{\leftarrow} \\
A & & A \wedge \top \\
& \top \wedge A & \\
\sigma_A^{\rightarrow} & \downarrow & \sigma_A^{\leftarrow} \\
A & & \top \wedge A \\
& \top & \\
r_x & \text{---} & t_{x,y,z} \\
& \text{cap} & \text{cup} \\
& x \leq y & x \leq y \wedge y \leq z \\
& \swarrow \quad \searrow & \swarrow \quad \searrow \\
& x \leq y & x \leq z
\end{array}$$

In the diagram for r_x we have a *cap* joining the two occurrences of x and in the diagram for $t_{x,y,z}$ we have a *cup* joining the two occurrences of y . We use an analogous terminology in other cases.

These diagrams and the function G on objects serve to define a functor G from M_{\leq} to the category Br of [9] (Section 2.3). Namely, G maps an arrow of M_{\leq} to an arrow of Br that corresponds to a diagram. The composition of M_{\leq} is mapped to composition in Br , which corresponds to vertical composition of diagrams, while the operation \wedge on the arrows of M_{\leq} is mapped to the operation of Br that corresponds to horizontal composition of diagrams (see [9], Section 2.3). We check by induction on the length of derivation that G is indeed a functor. Here is what we have in the basis of this induction for the specific equations of M_{\leq} :

$$\begin{array}{c}
(rt\delta): \\
\begin{array}{ccc}
\mathbf{1}_{x \leq y} \wedge r_y & \xrightarrow{x \leq y \wedge \top} & \delta_{x \leq y}^{\rightarrow} \\
\downarrow & \text{cup} & \downarrow \\
x \leq y \wedge y \leq y & & x \leq y \wedge \top \\
\text{---} & & \text{---} \\
t_{x,y,y} & & x \leq y
\end{array} \\
\\
(rt\sigma): \\
\begin{array}{ccc}
r_y \wedge \mathbf{1}_{y \leq x} & \xrightarrow{\top \wedge y \leq x} & \sigma_{y \leq x}^{\rightarrow} \\
\downarrow & \text{cup} & \downarrow \\
y \leq y \wedge y \leq x & & \top \wedge y \leq x \\
\text{---} & & \text{---} \\
t_{y,y,x} & & y \leq x
\end{array}
\end{array}$$

(tb):



(Note that there are no cups and caps in the diagrams corresponding to the left-hand sides of $(rt\delta)$ and $(rt\sigma)$; they were abolished by composing.)

So if $f = g$ in M_{\leq} , then $Gf = Gg$. Our purpose is to show also the converse for f and g of the same type; we show, namely, that G is a faithful functor from M_{\leq} to Br . We call this faithfulness result the *coherence* of M_{\leq} .

An arrow term of M_{\leq} of the form $f_n \circ \dots \circ f_1$, where $n \geq 1$, with parentheses tied to \circ associated arbitrarily, such that for every $i \in \{1, \dots, n\}$ we have that f_i is without \circ is called *factorized*. In a factorized arrow term $f_n \circ \dots \circ f_1$ the arrow terms f_i are called *factors*.

If β is a primitive arrow term of M_{\leq} which is not of the form $\mathbf{1}_B$, then β -terms are defined inductively as follows: β is a β -term; if f is a β -term, then for every object A of M_{\leq} we have that $\mathbf{1}_A \wedge f$ and $f \wedge \mathbf{1}_A$ are β -terms. In a β -term the subterm β is called the *head* of this β -term. For example, the head of the $t_{x,y,z}$ -term $(\mathbf{1}_{u \leq v} \wedge t_{x,y,z}) \wedge \mathbf{1}_{x \leq u}$ is $t_{x,y,z}$.

We define **1-terms** as β -terms by replacing β in the definition above by $\mathbf{1}_B$.

A factor that is a β -term for some β is called a *headed* factor. A factorized arrow term is called *headed* when each of its factors is either headed or a 1-term. A headed arrow term $f_n \circ \dots \circ f_1$ is called *developed* when f_1 is a 1-term and if $n > 1$, then every factor of $f_n \circ \dots \circ f_2$ is headed. Analogous definitions of factorized arrow term, factor, β -term, head, headed factor, headed factorized arrow term and developed arrow term can be given later for categories other than M_{\leq} , and we will not dwell on these definitions any more.

By using the categorial equations (*cat 1*) and (*cat 2*) and the bifunctionality equations we can easily prove by induction on the length of f the following lemma for M_{\leq} .

DEVELOPMENT LEMMA. *For every arrow term f there is a developed arrow term f' such that $f = f'$.*

An r -less arrow term of M_{\leq} is an arrow term of M_{\leq} in which r_x does not occur for any x . A headed factorized arrow term each of whose factors is an r_x -term for some x or a 1-term is called an r -factorized arrow term. We can easily prove the following lemma by applying the Development Lemma and the

equations $(rt\delta)$ and $(rt\sigma)$, besides bifunctionality, naturality and other obvious equations.

r-NORMALITY LEMMA. *For every arrow term f of M_{\leq} there is a headed factorized arrow term of M_{\leq} of the form $f_r \circ f'$ such that f is a developed r -less arrow term and f_r is an r -factorized arrow term and $f = f_r \circ f'$ in M_{\leq} .*

The arrow term $f_r \circ f'$ of this lemma is called the *r-normal form* of f .

Suppose $f, g: A \vdash B$ are arrow terms of M_{\leq} such that $Gf = Gg$. Let $f_r \circ f'$ and $g_r \circ g'$ be the r -normal forms of f and g respectively. Then $G(f_r \circ f') = G(g_r \circ g')$, and there is a bijection between the caps of $G(f_r \circ f')$ and $G(g_r \circ g')$. Moreover, there is a bijection between the caps in $G(f_r \circ f')$ and the r_x -factors of f_r , and analogously for $G(g_r \circ g')$ and g_r .

By using the bifunctionality equations we can achieve that f_r and g_r , which are r -factorized, are the same arrow term h . Since

$$\begin{aligned} G(h \circ f') &= Gh \circ Gf', \\ G(h \circ g') &= Gh \circ Gg', \\ G(h \circ f') &= G(h \circ g'), \end{aligned}$$

and Gh has no cups, it is easy to conclude that

$$Gf' = Gg'.$$

(In the category Br the arrow Gh , which has no cups, has a left inverse, which is its image in a horizontal mirror.)

There are no caps in Gf' and Gg' , and there is a bijection between the cups of Gf' and the $t_{x,y,z}$ -factors of f' , and analogously for Gg' and g' . A $t_{x,y,z}$ -factor of f' and a $t_{u,y,v}$ -factor of g' that correspond to each other according to these bijections are called *coupled*. Suppose Gf' and Gg' have at least one cup. Then by using the equation (tb) , besides the bifunctionality, naturality and other obvious equations, we obtain $f' = h_1 \circ f''$ and $g' = h_2 \circ g''$ where f'' and g'' are developed arrow terms, while h_1 is a $t_{x,y,z}$ -factor coupled with the $t_{u,y,v}$ -factor h_2 .

It is impossible that z coincides with v while x differs from u . Otherwise, the targets of h_1 and h_2 would differ. If x differs from u , and z differs from v , then in the source of f'' and g'' we would have variables occurring in the following order:

$$x \dots z \dots u \dots v \quad \text{or} \quad u \dots v \dots x \dots z,$$

with a cup between x and z in $G(h_1 \circ f'')$ corresponding to h_1 and a cup between u and v in $G(h_2 \circ g'')$ corresponding to h_2 . This is impossible because h_1 and h_2 are coupled. So $t_{x,y,z}$ coincides with $t_{u,y,v}$, and by using perhaps the bifunctionality equation $(\wedge 1)$ we can achieve that h_1 and h_2 are the same arrow term. From $G(h \circ f'') = G(h \circ g'')$ we conclude that $Gf'' = Gg''$.

Then we proceed by induction on the number of cups in Gf' , which is equal to Gg' , to show that $f' = g'$. In the basis of this induction we rely on Mac Lane's monoidal coherence (see [15] and [16], Section VII.2, or [8], Section 4.6). From that it follows that if for $f, g: A \vdash B$ arrow terms of M_{\leq} we have $Gf = Gg$, then $f = g$ in M_{\leq} . This proves the coherence of M_{\leq} .

Note that by omitting from the proof above the part involving r_x we would obtain an analogous coherence result for a category defined like M_{\leq} save that it lacks the arrows r_x and the specific equations $(rt\delta)$ and $(rt\sigma)$. We can also obtain coherence for a category defined like M_{\leq} but lacking the arrows $t_{x,y,z}$ and all the specific equations of M_{\leq} .

4 The coherence of S_{\leq}

The category S_{\leq} is defined like M_{\leq} save that we have for all formulae A and B the additional primitive arrow term

$$c_{A,B}: A \wedge B \vdash B \wedge A,$$

which is subject to the following additional equations:

naturality equation:

$$(c \text{ nat}) \quad (g \wedge f) \circ c_{A,B} = c_{D,E} \circ (f \wedge g),$$

specific equations of symmetric monoidal categories:

$$(cc) \quad c_{B,A} \circ c_{A,B} = \mathbf{1}_{A \wedge B},$$

$$(bc) \quad c_{A,B \wedge C} = b_{B,C,A}^{\rightarrow} \circ (\mathbf{1}_B \wedge c_{A,C}) \circ b_{B,A,C}^{\leftarrow} \circ (c_{A,B} \wedge \mathbf{1}_C) \circ b_{A,B,C}^{\rightarrow}.$$

The category S_{\leq} is a symmetric monoidal category (see [16], Section VII.7), in which \leq corresponds to a preordering relation.

We define the functor G from S_{\leq} to Br by extending the definition of G from M_{\leq} to Br with a clause corresponding to the following diagram:

$$\begin{array}{ccc} & A \wedge B & \\ c_{A,B} & \times & \\ & B \wedge A & \end{array}$$

That G is indeed a functor follows from well-known facts about symmetric monoidal categories (which were established in [15]; see also [8], Chapter 5). We can prove coherence for S_{\leq} with respect to G , i.e., we can prove that G is faithful, by imitating the proof of coherence for M_{\leq} in the preceding section. The only difference is that we appeal to symmetric monoidal coherence (see [15], Section 5, second edition of [16], Section XI.1, and [8], Section 5.3) where we appealed before to monoidal coherence, and we replace the proof that h_1 and

h_2 can be taken to be the same arrow term by the following alternative proof. An analogous proof could have already been used in the preceding section, but there, in the absence of $c_{A,B}$, we also had a slightly simpler argument.

For $f: A \vdash B$ an r -less arrow term of S_{\leq} , we say that a set U of occurrences of variables in A is *f -closed* when the following implication holds: if either $u \leq u'$ is a subformula of A or u and u' are connected by a cup of Gf , and one of u and u' is in U , then the other is in U too. It is easy to verify by induction on the complexity of f that

- (*) for every f -closed set U and for every atomic subformula $x \leq y$ of B , a member of U is connected by Gf to x if and only if a member of U is connected by Gf to y .

This holds in particular for f -closed sets generated by a single occurrence of a variable in A . We call such f -closed sets *maximal sequences*. It is easy to see that a maximal sequence is a set $\{u_1, u_2, \dots, u_{2n-1}, u_{2n}\}$ of occurrences of variables in A for $n \geq 1$ such that $u_{2i-1} \leq u_{2i}$ is a subformula of A , for $1 \leq i \leq n$, while u_{2j} and u_{2j+1} , for $1 \leq j \leq n-1$, are connected by a cup of Gf . Note that it follows from (*) that for a maximal sequence there must exist an atomic subformula $x \leq y$ of B such that u_1 in A is connected by Gf to the occurrence of x in $x \leq y$ in B and u_{2n} in A is connected by Gf to the occurrence of y in $x \leq y$ in B .

Suppose, as in the preceding section, that for r -less arrow terms $f', g': A \vdash B$ of S_{\leq} , we have $Gf' = Gg'$. Again, by using (tb), besides bifunctionality, naturality and other obvious equations, we obtain $f' = h_1 \circ f''$ and $g' = h_2 \circ g''$ where h_1 is a $t_{x,y,z}$ -factor coupled with the $t_{u,y,v}$ -factor h_2 . Then for the same maximal sequence u_1, \dots, u_{2n} in A we have that u_1 is connected by Gf' to the x of $t_{x,y,z}$ and by Gg' to the u of $t_{u,y,v}$. Analogously, we have that u_{2n} is connected by Gf' to the z of $t_{x,y,z}$ and by Gg' to the v of $t_{u,y,v}$. This means that $t_{x,y,z}$ coincides with $t_{u,y,v}$, and by using perhaps ($\wedge 1$) we can achieve that h_1 and h_2 are the same arrow term.

5 The coherence of M_{\equiv}

The category M_{\equiv} is defined like M_{\leq} save that \leq is replaced everywhere by \equiv , and we have for all variables x and y the additional primitive arrow term

$$s_{x,y}: x \equiv y \vdash y \equiv x,$$

which is subject to the following additional equations:

$$\begin{aligned} (ss) \quad & s_{y,x} \circ s_{x,y} = \mathbf{1}_{x \equiv y}, \\ (rs) \quad & s_{x,x} \circ r_x = r_x. \end{aligned}$$

The category M_{\equiv} is a monoidal category in which \equiv corresponds to an equivalence relation. Since the means of expression of M_{\equiv} are limited, this equivalence relation is an equality relation.

As the specific equations of M_{\leq} are parallel to the categorial equations (*cat 1*) and (*cat 2*) (see Section 2), so the equations (*ss*) and (*rs*) are parallel to equations of groupoids, i.e. categories where every arrow f has an inverse f^{-1} . The equation (*ss*) corresponds to $(f^{-1})^{-1} = f$ and (*rs*) corresponds to $1_A^{-1} = 1_A$.

We define the functor G from M_{\leq} to Br by extending G from M_{\leq} to Br with a clause corresponding to the following diagram:

$$\begin{array}{c} x \equiv y \\ \diagtimes \\ s_{x,y} \\ \diagtimes \\ y \equiv x \end{array}$$

Since for the equation (ss) we have

$$\begin{array}{ccc} x \equiv y & & x \equiv y \\ \diagup \quad \diagdown & & | \quad | \\ s_{x,y} & & 1_{x \equiv y} \\ & y \equiv x & x \equiv y \\ & \diagdown \quad \diagup & | \quad | \\ & s_{y,x} & x \equiv y \end{array}$$

and for (rs) we have

$$\begin{array}{ccc} \top & & \top \\ r_x & \text{---} \nearrow \curvearrowleft x \equiv x & r_x \\ s_{x,x} & \text{---} \searrow \curvearrowright x \equiv x & x \equiv x \end{array}$$

we can conclude that G is indeed a functor from M_{\equiv} to Br . In the remainder of this section we prove the faithfulness of G ; namely, the coherence of M_{\equiv} .

This proof is more complex than the other proofs of coherence in this paper. It involves a number of details. We will mention most of them, but not all, in order not to prolong the exposition excessively. We do not consider M_{\equiv} as the most significant category of this paper. (We find S_{\equiv} of the next section more important, and for it coherence is proved more easily.)

We can easily prove an analogue of the r -Normality Lemma of Section 3 for M_{\equiv} . To prove this analogue we apply also the equation (rs).

Then it is enough to prove coherence for r -less arrow terms of M_{\equiv} to obtain coherence for the whole of M_{\equiv} . (For r -factorized arrow terms we proceed as in Section 3.)

An arrow term of M_{\equiv} is called δ - σ -less when δ_A^{\rightarrow} , δ_A^{\leftarrow} , σ_A^{\rightarrow} or σ_A^{\leftarrow} does not occur in it for any A . We can establish the following.

δ - σ -NORMALITY LEMMA. For every r -less arrow term $f: A \vdash B$ of M_{\equiv} such that \top does not occur in $A \vdash B$ or both A and B are \top , there is a δ - σ -less arrow term $f': A \vdash B$ such that $f = f'$ in M_{\equiv} .

For the proof of this lemma we rely on bifunctionality and naturality equations, and on monoidal coherence. Intuitively, we push every δ_C^\rightarrow -factor in a headed factorized arrow term towards the right (or δ_C^\leftarrow -factor towards the left), where it or its descendant will disappear in virtue of the equations $(\delta\delta)$ or $(\sigma\sigma)$.

For every formula A we define a formula A^\dagger in which \top does not occur, or which is \top , in the following inductive manner: if A is atomic, then A^\dagger is A , and if A is $B \wedge C$, then $(B \wedge C)^\dagger$ is either $B^\dagger \wedge C^\dagger$ when neither B^\dagger nor C^\dagger is \top , or B^\dagger when C^\dagger is \top , or C^\dagger when B^\dagger is \top . It is clear that there is an isomorphism $\varphi_A: A \vdash A^\dagger$. For every arrow $f: A \vdash B$ of M_\equiv , let $f^\dagger: A^\dagger \vdash B^\dagger$ be the arrow $\varphi_B \circ f \circ \varphi_A^{-1}$. We have that $Gf = Gf^\dagger$, and $f = g$ if and only if $f^\dagger = g^\dagger$.

A type $A \vdash B$ is *diversified* when every variable in it occurs exactly twice (once in A and once in B , or twice in A , or twice in B). An arrow term whose type is diversified is also called *diversified*.

For every arrow term $f: A \vdash B$ of M_{\equiv} there is a diversified arrow term $f': A' \vdash B'$ of M_{\equiv} such that f is obtained from f' by substitution in the variables of f' . (Here variables are uniformly replaced by variables.) This is clear from Gf , which dictates how the diversification is to be achieved. If $f, g: A \vdash B$ are diversified arrow terms of M_{\equiv} , then $Gf = Gg$. We also have that for $f, g: A \vdash B$, there are diversified arrow terms $f', g': A' \vdash B'$ such that f and g are substitution instances of f' and g' respectively if and only if $Gf = Gg$.

For a headed factorized arrow term $f_n \circ \dots \circ f_1$ of M_{\equiv} , whose factors are f_1, \dots, f_n , we have that Gf_i , for $1 \leq i \leq n$, contains a crossing if and only if f_i is an $s_{x,y}$ -factor.

For $f: A \vdash B$ an arrow term of M_{\equiv} we say that a cup in the diagram corresponding to Gf *covers* an occurrence of \wedge in A when the ends of this cup are on different sides of this occurrence of \wedge . For example, in

$$\begin{array}{c}
 ((x \equiv y \wedge y \equiv z) \wedge x \equiv u) \wedge u \equiv v \\
 \diagdown \quad \curvearrowleft \quad \diagup \quad \diagup \quad \diagup \quad \diagup \quad \diagup \\
 ((x \equiv z \wedge x \equiv u) \wedge u \equiv v) \\
 \times \quad \diagup \quad \diagup \quad \diagup \quad \diagup \quad \diagup \quad \diagup \\
 (z \equiv x \wedge x \equiv u) \wedge u \equiv v \\
 \diagdown \quad \curvearrowleft \quad \diagup \quad \diagup \quad \diagup \quad \diagup \quad \diagup \\
 z \equiv u \wedge u \equiv v
 \end{array}$$

the y -cup covers only the leftmost occurrence of \wedge in $((x \equiv y \wedge y \equiv z) \wedge x \equiv u) \wedge u \equiv v$, and the x -cup covers the leftmost and middle occurrence of \wedge . The rightmost occurrence of \wedge is *uncovered*; i.e., it is not covered by any cup.

Suppose now that for the headed factorized arrow term $f: A \vdash B$ of M_{\equiv} we have that it is r -less, $\delta\sigma$ -less and diversified. Then there is an obvious one-to-one correspondence between occurrences of \wedge in B and uncovered occurrences of \wedge in A . There is also an obvious one-to-one correspondence between the following sets:

- the set of $t_{x,y,z}$ -factors of f ,
- the set of cups in Gf ,
- the set of variables occurring in A and not in B ,
- the set of occurrences of \wedge in A covered by a cup of Gf .

Note that all of these one-to-one correspondences that do not involve the first of these four sets do not depend on the arrow term f , but only on Gf .

An arrow term of M_{\equiv} is called *s-normal* when for every pair of variables (x, y) there is at most one occurrence of s in this arrow term with the indices x,y or y,x .

We can easily verify the following.

s-NORMALITY LEMMA. *For every diversified arrow term f of M_{\equiv} there is a developed s-normal arrow term f' of M_{\equiv} such that $f = f'$ in M_{\equiv} . If f is r -less, then f' is r -less too.*

This holds because, in a diversified developed arrow term, between two factors whose heads are $s_{x,y}$ or $s_{y,x}$ there can be no factor whose head is $t_{z,x,u}$ or $t_{z,y,u}$, which would be the only obstacle to bringing the two factors together, where they get cancelled.

In virtue of all that we have above it is enough to establish the following in order to prove coherence for M_{\equiv} .

AUXILIARY LEMMA. *Suppose f and g are developed, r -less, $\delta\sigma$ -less, diversified and s-normal arrow terms of M_{\equiv} of the same type. Then $f = g$ in M_{\equiv} .*

Proof. We proceed by induction on the number n of $s_{x,y}$ -factors and $t_{z,u,v}$ -factors in $f, g: A \vdash B$. This number must be the same in f and g because they are diversified and *s*-normal. (Note that $Gf = Gg$.) If $n = 0$, then we apply monoidal coherence. If $n > 0$, then there is in B an atomic subformula $x \equiv y$ such that either (1) y is in A on the left-hand-side of x , or (2) x is in A on the left-hand side of y and $x \equiv y$ is not a subformula of A .

In case (1) we have that

$$f = h \circ f' \quad \text{and} \quad g = h \circ g'$$

for an $s_{y,x}$ -factor h , and we may apply the induction hypothesis to f' and g' .

In case (2), we have that

$$f = h \circ f'$$

for a $t_{x,z,y}$ -factor h . There must be a $t_{u,z,v}$ -factor h' in g . Note that the occurrence of \wedge in A corresponding to h is covered just by the cup of Gf corresponding to h . This cup in Gg , which is equal to Gf , corresponds to h' in g . The arrow term g is of the form

$$g_m \circ \dots \circ g_1 \circ h' \circ g'$$

for $m \geq 0$ (if $m = 0$, then we have just $h' \circ g'$); here g_1, \dots, g_m are factors. We proceed by induction on m to show that g is equal to $h'' \circ g''$ for a $t_{x',z,y'}$ -factor h'' , which must be the same as h , for reasons given in the preceding section. (We only replace \leq by \equiv ; moreover, “ $u_{2i-1} \leq u_{2i}$ ” is replaced by “ $u_{2i-1} \equiv u_{2i}$ or $u_{2i} \equiv u_{2i-1}$ ” and “ $x \leq y$ ” is replaced by “ $x \equiv y$ or $y \equiv x$ ”.) Note that there can be no factor in $g_m \circ \dots \circ g_1$ whose head is $s_{u,v}$, because, as we said above, the occurrence of \wedge in A corresponding to h is covered just by the cup of Gf corresponding to h . \dashv

6 The coherence of S_{\equiv}

The category S_{\equiv} is defined like S_{\leq} save that, as when obtaining M_{\equiv} out of M_{\leq} , the symbol \leq is replaced everywhere by \equiv and we have the additional primitive arrow terms $s_{x,y}$ subject to the equation (ss) and (rs) of the preceding section, and the additional equation

$$(ts) \quad s_{x,z} \circ t_{x,y,z} = t_{z,y,x} \circ (s_{y,z} \wedge s_{x,y}) \circ c_{x \equiv y, y \equiv z}.$$

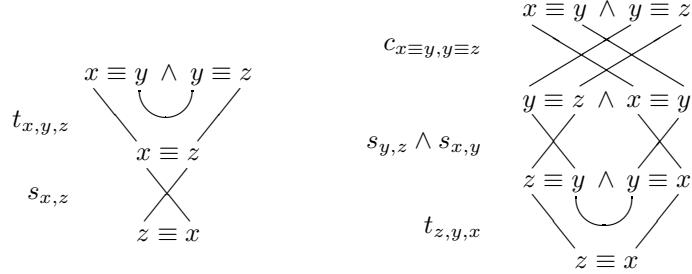
This equation is parallel to the following equation of groupoids (cf. Section 5): $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. It is analogous also to the equation $(\check{c}\check{w})$ of categories with coproducts (see [8], List of Equations, and cf. the end of Section 2 above).

Note that in the presence of (ts) we can derive (rt δ) from (rt σ), or vice versa. Here is a derivation of (rt δ) from (rt σ):

$$\begin{aligned} t_{x,y,y} \circ (\mathbf{1}_{x \equiv y} \wedge r_y) &= s_{y,x} \circ t_{y,y,x} \circ c_{y \equiv x, y \equiv y} \circ (s_{x,y} \wedge s_{y,y}) \circ (\mathbf{1}_{x \equiv y} \wedge r_y), \text{ with (ts),} \\ &= s_{y,x} \circ t_{y,y,x} \circ (r_y \wedge s_{x,y}) \circ c_{x \equiv y, \top}, \quad \text{with (rs) and (c nat),} \\ &= s_{y,x} \circ \sigma_{y \equiv x}^{\rightarrow} \circ (\mathbf{1}_{\top} \wedge s_{x,y}) \circ c_{x \equiv y, \top}, \quad \text{with (rt}\sigma\text{),} \\ &= \delta_{x \equiv y}^{\rightarrow}, \quad \text{with (}\sigma\text{ nat), (ss) and monoidal coherence.} \end{aligned}$$

The category S_{\equiv} is a symmetric monoidal category in which \equiv corresponds to an equivalence relation. Since the means of expression of S_{\equiv} are limited, this equivalence relation is an equality relation.

We define the functor G from S_{\equiv} to Br by combining what we had for G from M_{\equiv} to Br and for G from S_{\leq} to Br . Since for the equation (ts) we have



we can conclude that G is indeed a functor.

To prove the faithfulness of G , i.e. coherence for S_\equiv , we proceed in principle as for S_\leq in Section 4. Now that we have the equation (ts) , we can permute freely $t_{x,y,z}$ -factors with $s_{x,z}$ -factors, and eschew all the complications we had with M_\equiv in the preceding section.

7 Preorder, equivalence and adjunction

In this section we will show that assumptions concerning \leq and \equiv in categories of the preceding five sections amount to assumptions about some adjoint situations. This matter is related to matters considered in [14].

Let M_\leq^{-y} be the full subcategory of M_\leq in whose objects a particular variable y does not occur, and let M_\leq^y be the full subcategory of M_\leq whose objects are of the form $y \leq u \wedge A$ for y distinct from u and not occurring in A . Note that if the generating set \mathcal{V} of variables is infinite, then M_\leq^{-y} and M_\leq are isomorphic categories.

For every variable z distinct from y there is a functor F^z from M_\leq^{-y} to M_\leq^y defined as follows:

$$\begin{aligned} F^z A &=_{df} y \leq z \wedge A, \\ F^z f &=_{df} \mathbf{1}_{y \leq z} \wedge f. \end{aligned}$$

Conversely, there is a functor G^z from M_\leq^y to M_\leq^{-y} where

$$G^z(y \leq u \wedge A) =_{df} z \leq u \wedge A,$$

and $G^z f$ is obtained from the arrow term f by substituting z for y , i.e. by uniformly replacing y by z . (The function G^z on objects is also substitution of z for y , since y is distinct from u and does not occur in A .)

Let the arrow $\gamma_A^z: A \vdash z \leq z \wedge A$ of M_\leq^{-y} , whose target is $G^z F^z A$, be defined by

$$\gamma_A^z =_{df} (r_z \wedge \mathbf{1}_A) \circ \sigma_A^{\leftarrow},$$

and let the arrow

$$\varphi_{y \leq u \wedge A}^z : y \leq z \wedge (z \leq u \wedge A) \vdash y \leq u \wedge A$$

of M_{\leq}^y , whose source is $F^z G^z(y \leq u \wedge A)$, be defined by

$$\varphi_{y \leq u \wedge A}^z = df (t_{y,z,u} \wedge \mathbf{1}_A) \circ b_{y \leq z, z \leq u, A}^z.$$

Then we can verify easily by appealing to coherence for M_{\leq} that the functor F^z is left adjoint to G^z ; in this adjunction γ^z is the unit natural transformation, and φ^z the counit natural transformation (see [16], Section IV.1).

The “straightening of a sinuosity” involved in the equations $(rt\delta)$ and $(rt\sigma)$ (see the diagrams of Section 3) indicated that we have such an adjunction (cf. [3], Section 4.10, [9], Section 2.3, and [4], Section 7).

The arrow γ_A^z was defined in terms of r_z , but in M_{\leq}^{-y} we can define r_z in terms of γ^z as follows:

$$r_z = df \delta_{z \leq z}^z \circ \gamma_{\top}^z.$$

Analogously, the arrow $\varphi_{y \leq u \wedge A}^z$ was defined in terms of $t_{y,z,u}$, but we can define $t_{v,z,u}$ in M_{\leq}^{-y} in terms of φ^z as follows. Note first that in M_{\leq}^y we can take

$$t_{y,z,u} = df \delta_{y \leq u}^z \circ \varphi_{y \leq u \wedge \top}^z \circ (\mathbf{1}_{y \leq z} \wedge \delta_{z \leq u}^z)$$

for y different from z ; in M_{\leq}^{-y} we take

$$t_{v,z,u} = df \delta_{v \leq u}^z \circ G^v \varphi_{y \leq u \wedge \top}^z \circ (\mathbf{1}_{v \leq z} \wedge \delta_{z \leq u}^z).$$

Then the specific equations of M_{\leq} can be derived from the assumption that we have the adjunction above between M_{\leq}^{-y} and M_{\leq}^y , together with the equations

$$\gamma_A^z = ((\delta_{z \leq z}^z \circ \gamma_{\top}^z) \wedge \mathbf{1}_A) \circ \sigma_A^z,$$

$$\varphi_{y \leq u \wedge A}^z = ((\delta_{y \leq u}^z \circ \varphi_{y \leq u \wedge \top}^z \circ (\mathbf{1}_{y \leq z} \wedge \delta_{z \leq u}^z)) \wedge \mathbf{1}_A) \circ b_{y \leq z, z \leq u, A}^z.$$

These equations are obtained from the definition of γ_A^z in terms of r_z and the definition of $\varphi_{y \leq u \wedge A}^z$ in terms of $t_{y,z,u}$. They give a definition of γ_A^z for an arbitrary A in terms of γ_{\top}^z , and a definition of $\varphi_{y \leq u \wedge A}^z$ for an arbitrary A in terms of $\varphi_{y \leq u \wedge \top}^z$.

Note that a category equivalent to M_{\leq}^y is the full subcategory of M_{\leq} whose objects have a single occurrence of y as the leftmost variable.

The adjunction we had above between M_{\leq}^{-y} and M_{\leq}^y is obtained also when M_{\leq} is replaced by S_{\leq} , M_{\equiv} and S_{\equiv} . With M_{\equiv} we can take instead of the category M_{\equiv}^y , defined like M_{\leq}^y , the equivalent full subcategory M_{\equiv}^{y*} of M_{\equiv}

whose objects are those of the form $y \equiv u \wedge A$ or $u \equiv y \wedge A$ for y distinct from u and not occurring in A . We obtain an adjunction between M_{\equiv}^{-y} , defined like M_{\leq}^{-y} starting from M_{\equiv} , and M_{\equiv}^{y*} . In M_{\equiv}^{y*} we can define $s_{y,z}$ as follows:

$$s_{y,z} = df \quad \delta_{z \equiv y}^{\rightarrow} \circ \varphi_{z \equiv y \wedge \top}^z \circ (\mathbf{1}_{y \equiv z} \wedge \gamma_{\top}^z) \circ \delta_{y \equiv z}^{\leftarrow}.$$

With the categories S_{\equiv}^{-y} and S_{\equiv}^{y*} , defined analogously starting from S_{\equiv} , an analogous adjunction obtains. The category S_{\equiv}^{y*} is equivalent to the full subcategory of S_{\equiv} in whose objects y occurs exactly once.

8 The coherence of \dot{S}_{\leq}

Let us now suppose that terms are built with the help of a symbol \cdot that stands for a binary operation. We suppose, namely, that terms are not only variables, but for t_1 and t_2 terms we have that $t_1 \cdot t_2$ is a term. We use t, s, r, \dots , also with indices, for terms. To define atomic formulae we suppose that if t_1 and t_2 are terms, then $t_1 \leq t_2$ is an atomic formula, as well as \top . Formulae are defined otherwise as in Section 2.

The objects of the category \dot{S}_{\leq} are these new formulae; otherwise, \dot{S}_{\leq} is defined like S_{\leq} in Section 3 with the additional primitive arrow terms

$$a_{t_1, t_2, t_3, t_4}: t_1 \leq t_2 \wedge t_3 \leq t_4 \vdash t_1 \cdot t_3 \leq t_2 \cdot t_4$$

for all terms t_1, t_2, t_3 and t_4 , which are subject to the following additional equations:

$$(ra) \quad a_{t, t, s, s} \circ (r_t \wedge r_s) \circ \delta_{\top}^{\leftarrow} = r_{t \cdot s},$$

for $c_{A, B, C, D}^m = df \quad b_{A, C, B \wedge D}^{\rightarrow} \circ (\mathbf{1}_A \wedge (b_{C, B, D}^{\leftarrow} \circ (c_{B, C} \wedge \mathbf{1}_D) \circ b_{B, C, D}^{\rightarrow})) \circ b_{A, B, C \wedge D}^{\leftarrow}$:

$$(A \wedge B) \wedge (C \wedge D) \vdash (A \wedge C) \wedge (B \wedge D),$$

$$(ta) \quad a_{t_1, r_1, t_2, r_2} \circ (t_{t_1, s_1, r_1} \wedge t_{t_2, s_2, r_2}) = \\ t_{t_1 \cdot t_2, s_1 \cdot s_2, r_1 \cdot r_2} \circ (a_{t_1, s_1, t_2, s_2} \wedge a_{s_1, r_1, s_2, r_2}) \circ c_{t_1 \leq s_1, s_1 \leq r_1, t_2 \leq s_2, s_2 \leq r_2}^m.$$

The equation (ra) is parallel to the equation $(\wedge 1)$ of Section 2, and (ta) is parallel in the same manner to the equation $(\wedge 2)$. In another manner, the equation (ta) is analogous to the equation $(\check{b}\check{c}\check{w})$ of [8] (see the List of Equations).

We define the functor G from \dot{S}_{\leq} to the category Br by extending the definition of G from S_{\leq} to Br with a clause corresponding to the following diagram:

$$\begin{array}{ccc} & t_1 \leq t_2 \wedge t_3 \leq t_4 & \\ a_{t_1, t_2, t_3, t_4} & \left| \begin{array}{c} \diagup \\ \diagdown \end{array} \right. & \left| \begin{array}{c} \diagdown \\ \diagup \end{array} \right. \\ & t_1 \cdot t_3 \leq t_2 \cdot t_4 & \end{array}$$

(where each line stands for a family of parallel lines; cf. Section 3). To prove coherence for \dot{S}_\leq we use essentially the equation

$$\begin{aligned} a_{t_1,s_1,t_2,s_2} \circ (t_{t_1,r,s_1} \wedge \mathbf{1}_{t_2 \leq s_2}) = \\ t_{t_1 \cdot t_2, r \cdot s_2, s_1 \cdot s_2} \circ (a_{t_1,r,t_2,s_2} \wedge a_{r,s_1,s_2,s_2}) \circ c_{t_1 \leq r, r \leq s_1, t_2 \leq s_2, s_2 \leq s_2}^m \circ \\ \circ (\mathbf{1}_{t_1 \leq r \wedge r \leq s_1} \wedge ((\mathbf{1}_{t_2 \leq s_2} \wedge r_{s_2}) \circ \delta_{t_2 \leq s_2}^\leftarrow)), \end{aligned}$$

and another analogous equation with $t_{t_1,r,s_1} \wedge \mathbf{1}_{t_2 \leq s_2}$ on the left-hand side replaced by $\mathbf{1}_{t_1 \leq s_1} \wedge t_{t_2,r,s_2}$; we proceed otherwise using the ideas indicated in Section 3. This proof of coherence is parallel to the proof of the Development Lemma of Section 3; it formalizes the proof of this lemma on a different level.

When we take the category \dot{S}_\equiv obtained from S_\equiv as \dot{S}_\leq was obtained from S_\leq , with the additional equation

$$s_{t_1 \cdot t_2, s_1 \cdot s_2} \circ a_{t_1,s_1,t_2,s_2} = a_{s_1,t_1,s_2,t_2} \circ (s_{t_1,s_1} \wedge s_{t_2,s_2}),$$

which is parallel to $(f \wedge g)^{-1} = f^{-1} \wedge g^{-1}$, we can prove coherence analogously. In \dot{S}_\equiv the relation corresponding to \equiv is a congruence relation, which, due to the scarcity of the means of expression of \dot{S}_\equiv , is an equality relation.

To obtain coherence results for various categories extending \dot{S}_\equiv , which would formalize fragments of the equational theory of semigroups, or of the equational theory of monoids, commutative or not, we would need additional arrows analogous to the arrows $b_{A,B,C}^\rightarrow$, $b_{A,B,C}^\leftarrow$, δ_A^\rightarrow , δ_A^\leftarrow , σ_A^\rightarrow , σ_A^\leftarrow and $c_{A,B}$. For example, an arrow of type $\top \vdash t_1 \cdot (t_2 \cdot t_3) \equiv (t_1 \cdot t_2) \cdot t_3$ would correspond to b^\rightarrow . The additional equations for these new arrows would be parallel to the equations of monoidal or symmetric monoidal categories.

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